Lemma 0.1. *One consequence of receptivity: $\sigma : S \to A$ is a map of games. $x \overset{e}{-\!\!\!\subset} y$ then $\sigma x \overset{\sigma(e)}{-\!\!\!\subset} \sigma y$*

Proof. $y = x \cup \{e\}$ so $\sigma y = \sigma x \cup \{\sigma(e)\}$. If $\sigma(e) \in \sigma x$ then $\sigma x \subseteq^- \sigma y$ so by receptivity $\exists! x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ and $\sigma x' = \sigma y$. Either x or y fit the bill so $x = y$ and $e \in x$, a contradiction. □

Lemma 0.2. *This is also just a consequence of the axioms for a map. $\sigma : S \to A$ is a map of games. If $x \overset{e}{-\!\!\!\subset} y$ then $\sigma x \overset{\sigma(e)}{-\!\!\!\subset} \sigma y$*

Proof. $\sigma y = \sigma x \cup \{\sigma(e)\}$ and $\sigma(e) \in \sigma y$. If $\sigma(e) \in \sigma x$ then $\sigma(e) = \sigma(e')$ for some $e' \in x$ but then this contradicts injectivity of σ on y. □

Lemma 0.3. *If $z \overset{e}{-\!\!\!\subset} y$ and $z \overset{a}{-\!\!\!\subset} x$ in deterministic S and σ is receptive and $\sigma y \subseteq \sigma x$. Then $y = x$.*

Proof.

$$\sigma y = \sigma z \cup \{\sigma(e)\} \subseteq \sigma z \cup \{\sigma(a)\} = \sigma x$$

so $\sigma(e) = \sigma(a)$. If either $\mathrm{pol}(e) = +$ or $\mathrm{pol}(a) = +$ then by determinism, $x \overset{t}{-\!\!\!\subset}$ and $y \overset{t}{-\!\!\!\subset}$ so that $a, e \in t \in \mathcal{C}(S)$ so that by the property of a map of event structures (restriction to a configuration is injective) we have $a = e$ and consequently $x = y$. If, on the other hand $\mathrm{pol}(e) = \mathrm{pol}(a) = -$ then we can use receptivity: $\sigma z \subseteq^- \sigma x = \sigma y$ so

$$\exists! t \in \mathcal{C}(S) \;.\; z \subseteq t \wedge \sigma t = \sigma x$$

Both x and y fit the bill so $x = y$. □

Lemma 0.4. *For any configuration x there exists a covering chain*

$$\emptyset -\!\!\!\subset x_1 -\!\!\!\subset \ldots -\!\!\!\subset x_n = x$$

Proof. We show that for any two configurations x, y with $x \subseteq y$ we have a covering chain

$$x = x_1 -\!\!\!\subset \ldots -\!\!\!\subset x_n = x$$

by induction on $n = |y \setminus x|$. True for $n = 0$; the chain is $[x]$. Else choose a minimal element in $a_1 \in y \setminus x$. Then $x_1 \equiv x \cup \{a_1\}$ is a configuration; it is finite. It is consistent (as a subset of y). It is down-closed: $e' \leq e \in x_1$ then either $e \in x$ in which case $e' \in x$ by x being down-closed, or $e = a_1$ in which case $e' \leq a_1 \in y$ so $e' \in y$. If $e' \neq a_1$ and $e' \notin x$ then this contradicts minimality of a_1 in $y \setminus x$. □

Definition 0.5. For $e, e \in x \in \mathcal{C}(S)$

$$e \leq_x e' \equiv \forall y \subseteq x \;.\; (y \in \mathcal{C}(S) \wedge (e' \in y)) \Longrightarrow e \in y$$

Lemma 0.6. $e, e' \in x \in \mathcal{C}(S) \;.\; e \leq e' \Leftrightarrow e \leq_x e'$

Proof. Suppose $e \le e'$. Let $y \subseteq x$ and $y \in \mathcal{C}(S)$ and $e' \in y$ then $e \in y$ since y is down closed. Suppose $e \le_x e'$ and suppose that $e \not\le e'$. Then $[e'] \subseteq x$ (since x is down closed), it's a configuration, $e' \in [e']$ but $e \notin [e']$, which contradicts $e \le_x e'$. □

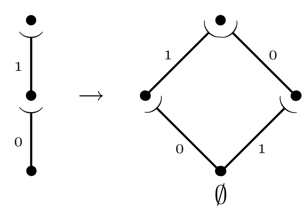
Lemma 0.7. *If $\sigma : S \to A$ is an event structure map. Then if $\sigma(e) \le_{\sigma x} \sigma(e')$ then $e \le_x e'$*

Proof. Let $y \subseteq x$ and $y \in \mathcal{C}(S)$ and $e' \in y$. Then $\sigma y \subseteq \sigma x$ and $\sigma y \in \mathcal{C}(A)$ and $\sigma(e') \in \sigma y$ so $\sigma(e) \in \sigma y$ so $\sigma(e) = \sigma(e'')$ for some $e'' \in y$. By injectivity of σ on x, we must have $e'' = e$ so $e \in y$ □

Lemma 0.8. *For $e, e' \in x \in \mathcal{C}(S)$ if $\sigma(e) \le \sigma(e')$ then $e \le e'$*

Proof. $\sigma(e) \le \sigma(e')$ so $\sigma(e) \le_{\sigma x} \sigma(e')$ so $e \le_x e'$ so $e \le e'$ □

The converse is not true. For example, take copy cat; the events are the same but there are extra dependencies in the domain event structure. Or



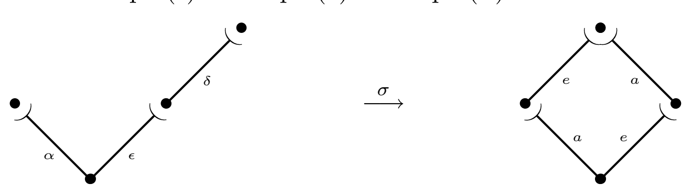
This reminds me of a continuous function (its inverse image map must preserve open sets, but its image map need not).

Lemma 0.9. *If $x, y \in \mathcal{C}(S)$ then $x \cap y \in \mathcal{C}(S)$*

Example 0.10.

$$\sigma(\alpha) = a \quad \sigma(\epsilon) = a \quad \sigma(\delta) = e$$

$$\mathrm{pol}(\epsilon) = - \quad \mathrm{pol}(\delta) = + \quad \mathrm{pol}(\alpha) = -$$



is a non-deterministic innocent map of games. The polarities of ϵ and δ are forced by innocence. And the polarity of α is then determined since $\mathrm{pol}(\alpha) = \mathrm{pol}(\sigma(\alpha)) = \mathrm{pol}(a) = \mathrm{pol}(\sigma(\epsilon)) = -$. It is not receptive though.

For a receptive example, simply take $A + A \to A$ with $A \equiv \emptyset \mathrel{-\!\!\!\subset} \{e\}$ and e positive.

Lemma 0.11. *Let $\sigma : S \to A$ be a deterministic strategy and $\sigma y \overset{\sigma(e)}{\mathrel{-\!\!\!\subset}} \sigma x$ for $x, y \in \mathcal{C}(S)$ and $e \in x$. Then $y \overset{e}{\mathrel{-\!\!\!\subset}} x$*

Proof. $\sigma(y) \subseteq \sigma(x)$ hence $y \subseteq x$. Hence $\sigma x \setminus \sigma y = \{\sigma(e)\}$. We have $x \setminus y = \{e\}$. Indeed, $e \in x$ but $e \notin y$. Suppose $e' \in x \setminus y$. If $\sigma(e') \in \sigma y$ then $\sigma(e') = \sigma(e'')$ for $e'' \in y$, implying $e' = e''$ by injectivity of σ on x, contradicting $e' \notin y$. Hence $\sigma(e') = \sigma(e)$, implying that $e' = e$.

Altogether we have $x \setminus y = \{e\}$ and $y \subseteq x$. Hence $y \overset{e}{\longrightarrow\!\!\subset} x$. □

I thought for a long time that a pairwise consistent set of events is consistent but this is plainly false. In particular, that if $x \overset{e}{\longrightarrow\!\!\subset}$ and $x \overset{e'}{\longrightarrow\!\!\subset}$ and $e, e' \in \mathrm{Con}$ then $x \cup \{e, e'\} \in \mathcal{C}(S)$. However, this is false as witnessed by the event structure

```
data ABC = A | B | C deriving (Ord,Eq)
abc = EventStructure {
        events = S.fromList [A,B,C],
        con = toSets [[],[A],[B],[C],[A,B],[A,C],[B,C]],
        ord = S.fromList [(A,A), (B,B), (C,C)]
}
```

I finally realized my mistake (which was an error of intuition about consistent sets) when I failed to prove this fact in Isabelle.

Example 0.12. A deterministic strategy is not always injective:

```
es1 = EventStructure {
        events = S.fromList [A,B,C,D],
        con = toSets [[],[A],[B],[C],[D],[A,B],[C,D]],
        ord = S.fromList [(A,A), (B,B), (C,C), (D,D), (A,B), (C,D)]
}

es2 = EventStructure {
        events = S.fromList [A,B,C],
        con = toSets [[],[A],[B],[C],[A,B],[B,C]],
        ord = S.fromList [(A,A), (B,B), (C,C)]
}

esm = EventStructureMap {
        dom = es1,
        cod = es2,
        sigma = \x -> case x of {A -> A; B -> B; C -> C; D -> B}
}
```

so it is not true that $\{e, e'\} \notin \mathrm{Con}$ implies that $\{f(e), f(e')\} \notin \mathrm{Con}$. The example arose from an attempt to generalize the proof that $x \cup \{e, e'\} \notin \mathrm{Con} \Longrightarrow (fx) \cup \{f(e), f(e')\} \notin \mathrm{Con}$ by induction and failing at a step which involved the case where the image events $\{f(e), f(e')\}$ are parallel. This example also illustrates that intersections are not preserved.

Lemma 0.13. *Let $f : S \to A$ be a deterministic strategy. Then* $\mathrm{Image}(f)$ *is a deterministic subfamily of* $\mathrm{Con}(A)$

Proof. Plus Innocence. Let

$$z, z \cup \{\alpha\}, z \cup \{\alpha, \beta\} \in \mathrm{Image} f$$

and $\mathrm{pol}(a) = +$ and $z \cup \{\beta\} \in \mathrm{Con}A$. So there exist $x_1, x_2, x_3 \in \mathrm{Con}S$ such that

$$z = \widehat{f}(x_1) \quad z \cup \{\alpha\} = \widehat{f}(x_2) \quad z \cup \{\alpha, \beta\} = \widehat{f}(x_3)$$

so there exists $e \in x_2, e' \in x_3$ such that $f(e) = \alpha$ and $f(e') = \beta$ hence

$$\widehat{f}(x_1) \xrightarrow{f(e)}\!\!\!\subset \widehat{f}(x_2) \xrightarrow{f(e')}\!\!\!\subset \widehat{f}(x_3)$$

so since f is deterministic, we have

$$x_1 \xrightarrow{e}\!\!\!\subset x_2 \xrightarrow{e'}\!\!\!\subset x_3$$

If not $x_1 \xrightarrow{e'}\!\!\!\subset$ then $e \rightarrow e'$ hence $f(e) \rightarrow f(e')$ since f is plus innocent (since $f(e) = \alpha$ then $\mathrm{pol}(e) = +$). But this contradicts $z \cup \{\beta\} \in \mathrm{Con}A$.

Minus Innocence. Similar.

Receptivity. □

Lemma 0.14. *Let F be a family of sets satisfying stability. If $[e]_x \subseteq y$ with $x, y \in F$ and $e \in x$ then $[e]_x = [e]_y$*

Proof. $[e]_x \subseteq [e]_y$. Suppose $a \in [e]_x$. Let $z \in F$, $z \subseteq y$, $e \in z$. Then

$$z \cap [e]_x \subseteq y \cap [e]_x \subseteq x$$

and since $z \uparrow^F [e]_x$ since $z, [e]_x \subseteq y$ then $z \cap [e]_x \in F$ and $e \in z \cap [e]_x$. So $a \in z \cap [e]_x$ so $a \in z$. Hence $a \in [e]_y$

$[e]_y \subseteq [e]_x$. Suppose $a \in [e]_y$. Let $z \in F$, $z \subseteq x$, $e \in z$. Then

$$z \cap [e]_x \subseteq x \cap [e]_x \subseteq y$$

and since $z \uparrow^F [e]_x$ since $z, [e]_x \subseteq x$ then $z \cap [e]_x \in F$ and $e \in z \cap [e]_x$. So $a \in z \cap [e]_x$ so $a \in z$. Hence $a \in [e]_x$ □

The following might be easier to do if we show that for a stable family F, and $p \in F$, $(\bigcup p, \mathbf{P}(p), \leq_p)$ is an event structure. In the following where we require a stable family, we don't appear to use coincidence-freeness.

Lemma 0.15. *Let F be a stable family of sets, and $p \in F$. Let $X \subseteq p$. Then $X \in F$ iff X is p-down-closed.*

Proof.

$$X = \bigcup_{e \in p} [e]_p$$

This is in F by completeness. (We know each $[e]_p \in F$ by the stability axiom and the intersection formula for $[e]_x$.) Conversely any $X \in F$ with $X \subseteq p$ is p-down-closed by definition of $\leq_p$. □

Lemma 0.16. *Let F be a stable family. Let $x \stackrel{a}{\longrightarrow\!\!\subset} \stackrel{b}{\longrightarrow\!\!\subset} \in F$ with $x \cup \{a,b\} \subseteq p \in F$. $x \stackrel{b}{\longrightarrow\!\!\subset} \stackrel{a}{\longrightarrow\!\!\subset} \in F$ iff $\neg a \leq_p b$*

Proof. If $a \leq_p b$ then $\neg(x \stackrel{b}{\longrightarrow\!\!\subset} \in F)$ since $a \notin x \cup \{b\} \subseteq p$.

Suppose $\neg(a \leq_p b)$. Then $x \cup \{b\}$ is down closed. Indeed, if $e \leq_p b$ ($e \in p$) then $e \in x \cup \{a,b\}$ (by definition of $\leq_p$) and also $e \neq a$ (otherwise $a \leq_p b$). Hence $e \in x \cup \{b\}$. Since $x \cup \{b\} \subseteq p \in F$ and $x \cup \{b\}$ is p-down closed then $x \cup \{b\} \in F$. We have $x \stackrel{b}{\longrightarrow\!\!\subset} \stackrel{b}{\longrightarrow\!\!\subset} \in F$ since we know that $x \cup \{a,b\}$ is p-down closed by $x \stackrel{a}{\longrightarrow\!\!\subset} \stackrel{b}{\longrightarrow\!\!\subset} \in F$. □

Lemma 0.17. *Let F be a stable family. Let $p \in F$ and $e, e' \in p$ and $e \lneq_p e'$. Suppose that for all $x \in F$, $\neg(x \stackrel{e}{\longrightarrow\!\!\subset} [e')_p \in F)$. Then there exists $a \in p$ such that $e \lneq_p a \lneq_p e'$.*

Proof. The intuition is that we push e as far as it will go up $[e')_p$ from $[e)_p$ (using lemma 0.16) until we hit a suitable a.

$$H_n \equiv \forall y \,.\, ((y \stackrel{e}{\longrightarrow\!\!\subset} \in F \wedge (y \cup \{e\}) \subset [e')_p \wedge |[e')_p \backslash y| = n) \implies \exists a \in p \,.\, e \lneq_p a \lneq_p e')$$

Once we have shown $\forall n \in \mathbf{N} \,.\, H_n$ then we take $y = [e)_p$ in $H_{|[e')_p \backslash [e)_p|}$ ($[e)_p \cup \{e\} \subset [e')_p$ since $[e)_p \subseteq [e')_p$ by transitivity of $\leq_p$ and equality doesn't hold (otherwise $[e)_p \stackrel{e}{\longrightarrow\!\!\subset} [e')_p \in F$)).

H_0 is true since if $|[e')_p \setminus y| = 0$ then $\neg(y \subset [e')_p)$.

Suppose H_n and $|[e')_p \setminus y| = n+1$ etc. Then $y \cup \{e\} \subset [e')_p$ (condition in H_{n+1}) so by chain there exists a such that $y \stackrel{e}{\longrightarrow\!\!\subset} \stackrel{a}{\longrightarrow\!\!\subset} \in F$ with $y \cup \{a,e\} \subseteq [e')_p$. If $e \leq_p a$ then we're done. Else $y \cup \{a\} \stackrel{e}{\longrightarrow\!\!\subset} \in F$ by lemma 0.16. We have $[e')_p \backslash (y \cup \{a\}) = ([e')_p \backslash y) \backslash \{a\})$ whose cardinality is n. And if $y \cup \{a,e\} = [e')_p$ then $y \cup \{a\} \stackrel{e}{\longrightarrow\!\!\subset} [e')_p$, which contradicts assumption. So we have $y \cup \{a,e\} \subset [e')_p$. Hence result by H_n. □

Lemma 0.18. *Let F be a family of sets with $e \in x \in F$ and $X \subseteq F$ and suppose that $X \cup \{[e]_x\}$ is $\subseteq$-down-closed in $\mathrm{Pr}(F)$. Then $(\bigcup X) \cup \{e\} = \bigcup(X \cup \{[e]_x\})$*

Proof. Suppose not. Then $\exists a \in [e]_x$ such that $a \notin (\bigcup X) \cup \{e\}$ so we have $[a]_x \subset [e]_x$ and since $X \cup \{[e]_x\}$ is down-closed then $[a]_x \in X$ so $a \in \bigcup X$, which is a contradiction. □

Lemma 0.19. *Let F be a stable family. If $X \stackrel{[e]_x}{\longrightarrow\!\!\subset}_{\mathrm{Pr}(F)}$ then $\bigcup X \stackrel{e}{\longrightarrow\!\!\subset} \in F$ and $(\bigcup X) \cup \{e\} = \bigcup(X \cup \{[e]_x\})$*

Proof. Since $X \cup \{[e]_x\} \in \mathrm{Con}_{\mathrm{Pr}(F)}$ then $y \equiv \bigcup(X \cup \{[e]_x\}) \in F$ so $[e]_x = [e]_y$. If $e \in \bigcup X$ then $e \in [e']_y$ for some $[e']_y \in X$ so $e \leq_y e'$ so $[e]_y \subseteq [e']_y$ and since X is $\subseteq$-down-closed then $[e]_y \in X$ hence $[e]_x \in X$, which is false.

And finally, $(\bigcup X) \cup \{e\} = \bigcup(X \cup \{[e]_x\})$ by lemma 0.18. □

Lemma 0.20. *Let F be a deterministic stable family with $F \subseteq \mathrm{Con}(A)$ for A an event structure. Then $\mathrm{ev} : \mathrm{Pr}(F) \to A$ is a deterministic strategy.*

Proof. Domain is an event structure. Let $[e]_x \in \Pr(F)$ for $e \in x \in F$ then $\{[e]_x\} \in \mathrm{Con}_{\Pr(F)}$ since $\bigcup\{[e]_x\} = [e]_x \in F$ (intersection formula). If $X \subseteq Y \in \mathrm{Con}_{\Pr(F)}$ then $\bigcup X \subseteq \bigcup Y \in F$ so $\forall x \in X \,.\, x \subseteq \bigcup Y \in F$ so $X \uparrow^F$ so $\bigcup X \in F$. Finally, if $x \subseteq y \in X \in \mathrm{Con}_{\Pr(F)}$ with $x \in \Pr(F)$ then $\bigcup X \in F$ and $\bigcup(X \cup \{x\}) = (\bigcup X) \cup x = \bigcup X \in F$ since $x \subseteq \bigcup X$ so $X \cup \{x\} \in \mathrm{Con}_{\Pr(F)}$.

Injective on Configurations. Let $X \in \mathrm{Con}_{\Pr(F)}$ and $[e]_x, [e']_x \in X$ (wlog we can take $x = \bigcup X$). Then if $\mathrm{ev}([e]_x) = \mathrm{ev}([e']_x)$ then $e = e'$ so clearly $[e]_x = [e']_x$.

Preserves Configurations. Let X be a configuration in $\Pr(F)$ (consistent, down closed). Then $\widehat{\mathrm{ev}}(X) = \bigcup X \in F$ (F is a set of configurations in A). Indeed, suppose $e \in \widehat{\mathrm{ev}}(X)$. Then $e = \mathrm{ev}([e']_y)$ for some $[e']_y \in X$ so $e = e'$ and $e \in [e]_y \subseteq \bigcup X$. Suppose $e \in \bigcup X$. Then $e \in [e']_{\bigcup X} \in X$. Thus $e \leq_{\bigcup X} e'$ so $[e]_{\bigcup X} \subseteq [e']_{\bigcup X}$ so by down-closure, $[e]_{\bigcup X} \in X$ so $e = \mathrm{ev}([e]_{\bigcup X}) \in \widehat{\mathrm{ev}}(X)$.

Innocence. Suppose $[e]_x \rightarrowtriangle_{\Pr(F)} [e]_y$ for $e \in x \in F$ and $e' \in y \in F$. Then $[e]_x \subseteq [e]_y$ so letting $p \equiv [e]_y$ we have $[e]_x = [e]_p$ and $[e]_y = [e]_p$. If there were no $x \in F$ such that $x \stackrel{e}{\longrightarrow}\!\!\subset [e')_p$ then by lemma 0.17, there exists a such that $e \lneq_p a \lneq_p e'$ so that $[e]_p \subset [a]_p \subset [e']_p$, contradicting $[e]_p \rightarrowtriangle_{\Pr(F)} [e]_p$. Therefore, we obtain such x and

$$x \stackrel{e}{\longrightarrow}\!\!\subset [e')_p \stackrel{e'}{\longrightarrow}\!\!\subset\in F$$

If $\neg(e \rightarrowtriangle_A e')$ then $x \stackrel{e'}{\longrightarrow}\!\!\subset$ in A. If either $\mathrm{pol}(e) = +$ (plus innocence) or $\mathrm{pol}(e') = -$ (negative innocence) then $x \stackrel{e'}{\longrightarrow}\!\!\subset\in F$ and $x \cup \{e'\} \subseteq p$ but $e \notin x \cup \{e'\}$ so $e \notin [e']_p$, which contradicts $[e]_x \rightarrowtriangle_{\Pr(F)} [e]_y$.

Deterministic. Suppose $X \stackrel{[e]_x}{\longrightarrow}\!\!\subset_{\Pr(F)}$ and $X \stackrel{[e']_{x'}}{\longrightarrow}\!\!\subset_{\Pr(F)}$ and $\mathrm{pol}(e) = +$. Then $\bigcup X \stackrel{e}{\longrightarrow}\!\!\subset\in F$ and $\bigcup X \stackrel{e'}{\longrightarrow}\!\!\subset\in F$ by lemma 0.19. So since F is a deterministic family then $\bigcup X \cup \{e, e'\} \in F$. We have

$$\begin{aligned}\bigcup(X \cup \{[e]_x, [e']_{x'}\}) &= (\bigcup(X \cup \{[e]_x\})) \cup (\bigcup(X \cup \{[e']_{x'}\})) \\ &= ((\bigcup X) \cup \{e\}) \cup ((\bigcup X) \cup \{e'\}) = (\bigcup X) \cup \{e, e'\}\end{aligned}$$

using lemma 0.19. It remains to show that $X \cup \{[e]_x, [e']_{x'}\}$ is $\subseteq$-down-closed in $\Pr(F)$. This follows since the union of two down-closed sets is down-closed.

Receptive. Suppose $\widehat{\mathrm{ev}}(X) \stackrel{a}{\longrightarrow}\!\!\subset_A$ for X a configuration in $\Pr(F)$ and $\mathrm{pol}(a) = -$. Then from above (see preserves configurations), $\widehat{\mathrm{ev}}(X) = \bigcup X \in F$. By receptivity for F we have that $y \equiv (\bigcup X) \cup \{a\} \in F$. We have $X \cup \{[a]_y\} \in \mathrm{Con}_{\Pr(F)}$ is down-closed in $\Pr(F)$. Indeed, suppose we have $[e]_y \subseteq [a]_y$ for some $e \in y$ with $[e]_y \notin X$. If $e \in \bigcup X$ then $[e]_y \in X$ (since $\bigcup X \in F$ and $\bigcup X \subseteq y$ and by definition of $[e]_x$) so $e \notin \bigcup X$ so by definition of y we have $e = a$. We have $X \cup \{[a]_y\} \in \mathrm{Con}_{\Pr(F)}$ by lemma 0.18. Since we have $a \notin \bigcup X$ then $[a]_y \notin X$. Altogether we have $X \stackrel{[a]_y}{\longrightarrow}\!\!\subset_{\Pr(F)}$. Uniqueness. If $X \stackrel{[a]_{x'}}{\longrightarrow}\!\!\subset_{\Pr(F)}$ then by lemma 0.18

we have $y = (\bigcup X) \cup \{a\} = \bigcup(X \cup \{[a]_{x'}\})$ so $[a]_{x'} \subseteq y$ so $[a]_{x'} = [a]_y$ by lemma 0.14 $\square$

Thought: As event structures progress, information increases. This sounds like sigma algebra, martingale theory area.

Definition 0.21 (parallel condition)**.** A family of sets F satisfies the parallel condition iff whenever $x \stackrel{a}{-\!\!\!\subset}\in F$ and $x \stackrel{b}{-\!\!\!\subset}\in F$ and $x \cup \{a, b\} \subseteq p \in F$ then $x \cup \{a, b\} \in F$.

Definition 0.22 (intersection condition)**.** A family of sets F satisfies the intersection condition iff whenever for $x, y \in F$ with $x, y \subseteq p \in F$ we have $x \cap y \in F$.

Lemma 0.23. *Let F be a family of finite sets satisfying chain and parallel. Let $x, y \in F$ and $p \in F$ and $x \subseteq y \subseteq p$ and $x \stackrel{e}{-\!\!\!\subset}\in F$ with $x \cup \{e\} \subseteq p$. Then $y \cup \{e\} \in F$.*

Proof.

$$H_n \equiv \forall x\ .\ (x \subseteq y \wedge x \stackrel{e}{-\!\!\!\subset}\in F \wedge x \cup \{e\} \subseteq p \wedge |y \setminus x| = n) \Longrightarrow y \cup \{e\} \in F$$

For $n = 0$ we have $y = x$ so $x \cup \{e\} \in F$ since $x \stackrel{e}{-\!\!\!\subset}\in F$.

Assume H_n and the assumptions of H_{n+1}. Since $|y \setminus x| = n + 1$ then $x \subset y$ so by chain, we have a such that $x \stackrel{a}{-\!\!\!\subset}\in F$ with $a \in y \setminus x$. Since $x \cup \{a, e\} \subseteq p$ then by parallel, $x \cup \{a, e\} \in F$. If $a = e$ then we're done since then $e \in y$ else $x \cup \{a\} \stackrel{e}{-\!\!\!\subset}\in F$ and we use H_n. $\square$

Lemma 0.24. *If F is a family of finite sets satisfying the chain and parallel conditions and contains $\emptyset$ then if $z_1 \uparrow^F z_1$ then $z_1 \cup z_2 \in F$.*

Proof. Take

$$H_n \equiv \forall z_1, z_2, x \subseteq p, \in F\ .\ ((x \subseteq z_1, z_2 \wedge |z_1 \setminus x| = n) \Longrightarrow z_1 \cup z_2 \in F)$$

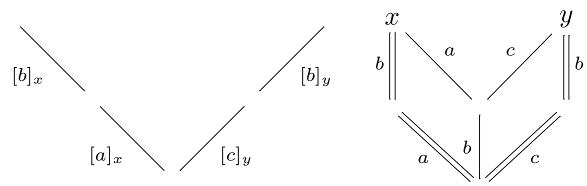
If $n = 0$ then since $x \subseteq z_1$, $z_1 = x$. So $z_1 \subseteq z_2$ so $z_1 \cup z_2 = z_2 \in F$.

Suppose H_n. Let $z_1, z_2, x \subseteq p, \in F$ and suppose $x \subseteq z_1, z_2$ and $|z_1 \setminus x| = n+1$. Then by chain we have $a \in z_1$ such that $x \stackrel{a}{-\!\!\!\subset}\in F$. If $|z_2 \setminus x| = 0$ then we're done already since $z_2 \subseteq z_1$. Else we can find $b \in z_2$ such that $x \stackrel{b}{-\!\!\!\subset}\in F$. By parallel, we have $x \cup \{a, b\} \in F$. Now take $z_1' = z_1 \cup \{b\}$ and $z_2' = z_2 \cup \{a\}$ and $x' = x \cup \{a, b\}$. We have $x' \subseteq z_1', z_2'$ and we have $z_1', z_2' \in F$ using lemma 0.23. We have $|z_1' \setminus x'| = n$. If $a = b$ then $z_1' = z_1$ since $a \in z_1$; else $|z_1'| = |z_1| + 1$ and $|x'| = |x| + 2$ so since $x' \subseteq z_1'$ and $x \subseteq z_1$ then

$$|z_1' \setminus x'| = |z_1| + 1 - (|x| + 2) = |z_1 \setminus x| - 1 = n$$

$\square$

Example 0.25. Here is the event structure map for the deterministic sub family of example 0.12



Here, $x = \{a, b\}$ and $y = \{c, b\}$. On the left is $\mathrm{Pr}(F)$ and on the right is A with $F = \{\emptyset, \{a\}, \{c\}, x, y\}$

We need coincidence freeness to show that $\mathrm{ev}([e]_x) = e$. This follows from the chain condition.

Lemma 0.26. *Let F be a family of sets satisfying chain and containing $\emptyset$. Let $x \in F$ with $x \neq \emptyset$ then there exists y and e such that $y \stackrel{e}{\longrightarrow\!\!\subset} x \in F$.*

Lemma 0.27. *Let F be a family of sets satisfying chain and containing $\emptyset$. Then F is coincident free*

Proof. i.e. Let $e, e' \in x \in F$ with $e \neq e'$. Then there exists $y \in F$ such that

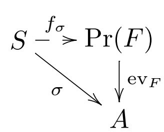
$$e, e' \notin y \wedge y \subseteq x \wedge y \cup \{e\} \in F$$

Take

$$H_n \equiv \forall e, e', z \,.\, (e, e' \in z \wedge e \neq e' \wedge z \subseteq x \wedge |z| = n) \implies \exists y \,.\, y \subseteq z \wedge e \in y \Leftrightarrow e' \notin y$$

□

Lemma 0.28. *If $\sigma : S \to A$ is any deterministic strategy with image F then there is a unique strategy $f_\sigma : S \to \mathrm{Pr}(F)$ (given by $f_\sigma(s) \equiv [\sigma(s)]^F_{\sigma^\to([s]^S)}$) such that this diagram commutes:*



Proof. sketch. preserves configurations. Let $x \in \mathcal{C}(S)$ then

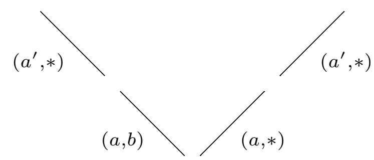
$$f_\sigma{}^\to(x) = \{[\sigma(s)]^F_{\sigma^\to(x)} \mid s \in x\} = \{[t]^F_{\sigma^\to(x)} \mid t \in \sigma^\to(x)\} \in \mathcal{C}(\mathrm{Pr}(F))$$

innocent. If $s \rightarrowtriangle s'$ and we're in the case of +/- innocence then $\sigma(s) \rightarrowtriangle \sigma(s')$ so $[\sigma(s)]^F_{\sigma^\to([s]^S)} = [\sigma(s)]^F_{\sigma^\to([s']^S)} \rightarrowtriangle_{\mathrm{Pr}(F)} [\sigma(s')]^F_{\sigma^\to([s']^S)}$ (we have $[\sigma(s)]^F_{\sigma^\to([s']^S)} \subseteq [\sigma(s')]^F_{\sigma^\to([s']^S)}$; if not then there exists $[\sigma(s)]^F_{\sigma^\to([s']^S)} \subset [a]^F_{\sigma^\to([s']^S)} \subset [\sigma(s')]^F_{\sigma^\to([s']^S)}$ for some $a \in \sigma^\to([s']^S)$ so $a = \sigma(t)$ for $t \lneq s'$ and since $\sigma(s) \lneq_{\sigma^\to([s']^S)} \sigma(t)$ then $s \lneq_{[s']^S} t$ so $s \lneq t$, contradicting $s \rightarrowtriangle s'$).

receptive. Suppose $f_\sigma{}^{\rightarrow}(x) \overset{[a]^F_y}{\longrightarrow\!\!\subset}_{\mathrm{Pr}(F)}$. We have $\bigcup f_\sigma{}^{\rightarrow}(x) = \sigma^{\rightarrow}(x)$ so $\sigma^{\rightarrow}(x) \overset{a}{\longrightarrow\!\!\subset}_A$ by lemma 0.19 so by receptivity of σ there exists α such that $x \overset{\alpha}{\longrightarrow\!\!\subset}_S$ and $\sigma(\alpha) = a$ so $f_\sigma(\alpha) = [a]^F_y$. And if $x \overset{\alpha'}{\longrightarrow\!\!\subset}$ and $f_\sigma(\alpha') = [a]^F_y$ then $[a]^F_y = [\sigma(\alpha')]_{\sigma^{\rightarrow}([\alpha'])} \in F$ (since F is stable family) so $[a]^F_y = \sigma^{\rightarrow}(z)$ for some $z \in \mathcal{C}(S)$ so $[a]^F_y = [a]^F_{\sigma^{\rightarrow}(z)} = [\sigma(\alpha')]^F_{\sigma^{\rightarrow}(z)}$ so $\sigma(\alpha') = a$ so $\alpha' = \alpha$ by receptivity of σ. □

Example 0.29. Why a product of prime stable families is not always prime. Suppose $a \leq_1 a'$ and $b \leq_2 b'$. Let $x \equiv \{(*, b), (a', b')\}$ and $y \equiv \{(a, *), (a', b')\} \in F$. Then $(a', b') \in x \cap y$ but $[(a', b')]_x = x$ and $[(a', b')]_y = y$.

Another example:



Definition 0.30. Let `F :: Set (Set a)` be a stable family; as a list, order it by size of set and get `configs`. Let `es` be a `Map a [(Int,Set a)]` and `ces` be a `Map (Set a, a) Int`

```
for c in configs
  for e in c
      case lookup e es of
         Nothing -> (add (e |-> [(0,c)]) to es) &&
                    (add ((c,e) |-> 0) to ces)
         Just el -> case (first in el s.t. c' <= c) of
                       Just (i,c') -> (add ((c,e) |-> i) to ces)
                       Nothing -> let (i,_) be (last in el)
                                  in (add (e |-> el ++ [(i+1,c)]) to es) &&
                                     (add ((c,e) |-> i+1) to ces)
```

Using `ces` and `es` we define an event structure. First, we lift to the data type `Pr` using `ces`

```
data Pr a = Pr a Int
```

The new configurations are

```
einc c e = let (Just i) = M.lookup ces (c,e) in Pr e i
configs' = L.map (\c -> S.map (\e -> einc c e) c) configs
```

`events` is the union of `configs'`, the consistent sets subsets of `configs'` and the order is

```
ltconfig (e,(i,c)) = S.map (\e' -> (einc c e', (Pr e i))) c
S.unions (L.map ltconfig (M.toList es))
```

Define `mu :  Pr e i |-> e` and $\texttt{pr}(F)$ as the event structure formed here.

Lemma 0.31. *Definition 0.30 is an alternate definition of* Pr *(i.e.* `pr` *is an adjunction with* `mu` *as counit).*

Proof. First, we need to show that the structure defined in definition 0.30 gives an event structure.

The integers label the minimal configurations that each event is in. If `Pr e i` is an event then there is a unique `c` in `F` such that

```
S.elem (e,(i,c)) (S.fromList (M.toList es))
```

(i.e. `i` labels configuration `c`, a minimal configuration with `e` in), so we can define a function $f : \texttt{Pr e i} \mapsto [\texttt{e}]_{\texttt{c}}$. We need to show that f is an isomorphism of event structures (from `pr(F)` to $\Pr(\texttt{F})$). Then it follows that `mu` is counit since $\texttt{mu} = \max \circ f$. The inverse of f is given as follows. Suppose that $[\texttt{e}]_x$ is an element of $\Pr(\texttt{F})$ then there exists a unique `(i,c)` such that $\texttt{c} \subseteq x$ and

```
S.elem (e,(i,c)) (S.fromList (M.toList es))
```

□

Definition 0.32 (Demand Map)**.** Let $\sigma : S \to A$ be a map of games. Define $d_\sigma : \mathcal{C}(S^+) \to \mathcal{C}(A)$ where $S^+ \equiv S{\downarrow}\{e \in S \mid \mathrm{pol}(e) = +\}$ by

$$d_\sigma x \equiv \sigma[x]_S$$

Lemma 0.33. *Let U, A, S be games. Let $\sigma : S \to A$ and $f : U \to A$ be maps of games with σ receptive and $g : U \rightharpoonup S^+$ and suppose for all $x \in \mathcal{C}(U)$, $d_\sigma(gx) \subseteq^- fx$. Then there exists a unique map of games ϕ with $\sigma \circ \phi = f$ and for all $x \in \mathcal{C}(U)$, $[gx]_S \subseteq^- \phi x$.*

Proof. Define $p : \mathcal{C}(U) \to \mathcal{C}(S)$ by $p(x) \equiv [gx]_S$. Then p is monotonic since it is the composite of monotonic functions and $\sigma p(x) = d_\sigma(gx) \subseteq^- fx$ by assumption for $x \in \mathcal{C}(U)$. The result follows from lemma 22 in the paper. □

Lemma 0.34. *Let $\sigma, \sigma' : S \to A$ be receptive maps of games. If $d_\sigma = d_{\sigma'}$ then there exists an isomorphism of games $\phi : S \to S$ such that $\sigma = \sigma' \circ \phi$*

Proof. Define $q : S \rightharpoonup S^+$ to be identity on plus events. We have

$$d_\sigma(qx) = d_\sigma(x^+) = d_{\sigma'}(x^+) = \sigma'[x^+]_S \subseteq^- \sigma' x$$

so by lemma 0.33 there exists an event structure map $\phi : S \to S$ such that $\sigma' = \sigma \circ \phi$ and by symmetry, a map θ such that $\sigma = \sigma' \circ \theta$. Hence

$$\sigma' \circ (\theta \circ \phi) = \sigma \circ \phi = \sigma'$$

so by lemma 0.33 $\theta \circ \phi = 1_S$ and similarly $\phi \circ \theta = 1_S$ □