Definition 0.1. Let S be a game. Define

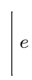
$$S^+ \equiv S \downarrow \{e \in S \mid \mathrm{pol}(e) = +\}$$

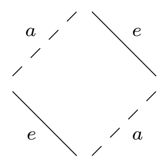
Definition 0.2. Let S, A be games and suppose $d : \mathcal{C}(S) \to \mathcal{C}(A)$ then we say that d preserves unions iff whenever $x, y \in \mathcal{C}(S)$ and $x \cup y \in \mathcal{C}(S)$ then $d(x \cup y) = d(x) \cup d(y)$

Definition 0.3. Let S, A be games and $\sigma : S \to A$ a map of games. Define $d_\sigma : \mathcal{C}(S^+) \to \mathcal{C}(A)$ by

$$d_\sigma(y) \equiv \sigma^{\to}([y]^S)$$

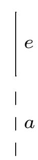
Example 0.4. A dashed line indicates negative polarity:

$E \equiv$ 

$A \equiv$ 

Define $d : \mathcal{C}(E) \to \mathcal{C}(A)$ by $d : \emptyset \mapsto \emptyset$ and $d : \{e\} \mapsto \{e\}$. In the following cases, suppose $\sigma : S \to A$ is given by $\sigma : a \mapsto a$ and $\sigma : e \mapsto e$.

Case 1.

$S \equiv$ 

Then σ is a receptive, innocent map of games satisfying

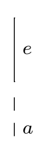
$$\forall x \in \mathcal{C}(S) \; . \; d(x^+) \subseteq^- \sigma^{\to}(x) \qquad (\dagger)$$

but not

$$\forall y \in \mathcal{C}(S^+) \; . \; d(y) = \sigma^{\to}([y]_S) \qquad (\ddagger)$$

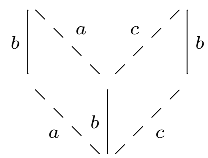
Case 2. $S = A$. Then σ is a receptive, innocent map of games satisfying both † and ‡.

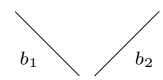
If d satisfies ‡ then for each positive event, d specifies the negative events upon which it depends in S. If $d(\{e\}) = \{a, e\}$ then this determines S as

$S \equiv$ 

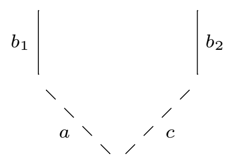
The only alternative is that $d(\{e\}) = \{e\}$; in this case, S is uniquely determined as A (under the requirement that σ is receptive), which is case 2.

Example 0.5. One way in which S_d could be built is as a sequence of stable families using d satisfying suitable conditions. This shows that in general the resulting stable family would not always be prime (since σ may not be injective):

$A \equiv$ 

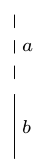
$E \equiv$ 

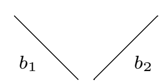
and $d(\{b_1\}) = \{a, b\}$ and $d(\{b_2\}) = \{c, b\}$. Then the unique receptive map $\sigma : S \to A$ is given by

$S \equiv$ 

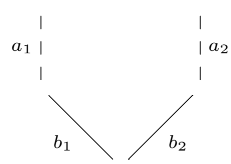
with $\sigma = [a \mapsto a, c \mapsto c, b_1 \mapsto b, b_2 \mapsto b]$.

Example 0.6. This example shows how σ may not be injective in a different way; let $d(\{b_1\}) = d(\{b_2\}) = \{b\}$ satisfy ‡ and

$A \equiv$ 

$E \equiv$ 

Then receptive $\sigma : S \to A$ is determined as

$S \equiv$ 

with $\sigma = [a_1 \mapsto a, a_2 \mapsto a, b_1 \mapsto b, b_2 \mapsto b]$. This example is different from the last in that branching happens at the positive event and everything above is "copied" from A and attached onto each branch in S.

Definition 0.7. Let E, A be games with all events in E positive. Suppose $d : \mathcal{C}(E) \to \mathcal{C}(A)$. Define

$$f_d \equiv \{(a, \alpha) \in E_E \times A_E^+ \mid \forall y \in \mathcal{C}(E) \;.\; y \xrightarrow{a}\!\!\subset_E \Longrightarrow \\ ((d(y) \subseteq^- d(y \cup \{a\}) \setminus \{\alpha\} \in \mathcal{C}(A)) \wedge (\alpha \in (d(y \cup \{a\}) \setminus d(y))))\}$$

Suppose further that $f_d : E_E \to A_E^+$. Define the "visible" negative events

$$M_d \equiv \bigcup\{x^- \mid (x \in \mathcal{C}(A)) \wedge (\exists y \in \mathcal{C}(E) \,.\, d(y) \subseteq^- x)\}$$

Define $P_d \equiv E_E$ and

$$N_d \equiv \{(x, a) \in \mathrm{Con}_E \times M_d \mid (f_d{}^{\to}(x) = ([a]^A)^+) \wedge (d([x]^E) \cup \{a\} \in \mathrm{Con}_A)\}$$

Define $\sigma_d : (P_d \uplus N_d) \to A_E$ by

$$\sigma_d(b) \equiv f_d(b) \qquad \sigma_d((x, a)) \equiv a$$

and, for $R \subseteq (S_d)_E$,

$$\chi_d(R) \equiv \{b \mid b \in R\} \cup \bigcup\{x \mid \exists a \,.\, (x, a) \in R\}$$

Now we define a game structure S_d by $(S_d)_E \equiv P_d \uplus N_d$ and

$$\begin{aligned}\leq'_{S_d} \equiv &\{(b_1, b_2) \in P_d \times P_d \mid b_1 \leq_E b_2\} \cup \\ &\{(b, (x, a)) \in P_d \times N_d \mid b \in [x]^E\} \cup \\ &\{((x_1, a_1), (x_2, a_2)) \in N_d \times N_d \mid (x_1 \subseteq x_2) \wedge (a_1 \leq_A a_2)\} \cup \\ &\{((x, a), b) \in N_d \times P_d \mid a \in d([b]^E) \wedge (x \subseteq [b]^E)\}\end{aligned}$$

and

$$\leq_{S_d} \equiv \leq'_{S_d} \cup \{((x_1, a_1), (x_2, a_2)) \mid \exists b \in P_d \,.\, ((x_1, a_1), b), (b, (x_2, a_2)) \in \leq'_{S_d}\}$$

$$\mathrm{Con}_{S_d} \equiv \{X \subseteq (S_d)_E \mid (\chi_d([X]^{S_d}) \in \mathcal{C}(E)) \wedge (\sigma_d{}^{\to}([X]^{S_d}) \in \mathcal{C}(A))\}$$

$$\mathrm{pol}_{S_d}(b) \equiv + \qquad \mathrm{pol}_{S_d}((x, a)) \equiv -$$

Finally, define $\phi_d : E \to S_d^+$ to be the injection from E_E into the left component of $(S_d)_E$.

Definition 0.8. Let A and E be games such that the events of E are all positive. Let $d : \mathcal{C}(E) \to \mathcal{C}(A)$. Then call the following formula $\beta(d, E, A)$

$$\begin{aligned}&\forall a \in E_E \exists \alpha \in A_E^+ \,.\, \forall y \in \mathcal{C}(E)(y \xrightarrow{a}\!\!\subset_E \Longrightarrow \\ &\quad ((d(y) \subseteq^- d(y \cup \{a\}) \setminus \{\alpha\} \in \mathcal{C}(A)) \wedge (\alpha \in (d(y \cup \{a\}) \setminus d(y)))))\end{aligned}$$

Example 0.9. Let A have events $\{p_0, p_1, n_0\}$ with $\mathrm{pol}(p_i) = +$ and $\mathrm{pol}(n_i) = -$ with no inconsistency and trivial order (so configurations are at the vertices of a cube). Let $E = S \downarrow \mathrm{pol}^{\leftarrow}(-)$. Let $d = [\emptyset \mapsto \emptyset, \{p_0\} \mapsto \{p_0\}, \{p_1\} \mapsto \{p_1\}, \{p_0, p_1\} \mapsto \{n_0, p_0, p_1\}]$. Then d satisfies $\beta(d, E, A)$. And σ_d is the identity but this same map is generated by $d' = d[\{p_0, p_1\} \mapsto \{p_0, p_1\}]$. To remedy this, we can require that d preserve unions, which means that if $\{p, p'\}$ depends on n then either p or p' must have depended on it before they joined up together. This is a natural requirement since certainly, for any σ, d_σ preserves unions

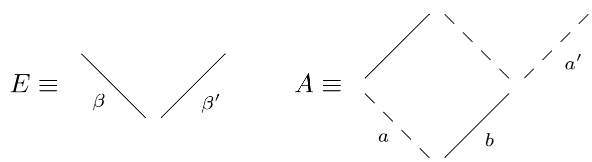
$$\sigma^{\to}([y \cup y']^S) = \sigma^{\to}([y]^S \cup [y']^S) = \sigma^{\to}([y]^S) \cup \sigma^{\to}([y']^S)$$

Example 0.10. Let $A = \{p_0, p_1, n_0, n_1\}$ with $\{p_0, p_1, n_0, n_1\}$ the only inconsistent subset, and trivial order. Let $E = \{p_0, p_1\}$ with $p_0 \leq_E p_1$. And define $d = [\emptyset \mapsto \emptyset, \{p_0\} \mapsto \{p_0\}, \{p_0, p_1\} \mapsto \{p_0, p_1\}]$. Let $X = \{(\emptyset, n_0), (\emptyset, n_1), p_1\}$. Then $\chi_d(X) = \{p_1\}$ and $\sigma_d{}^{\rightarrow}(X) = \{n_0, n_1, p_1\}$. If we had

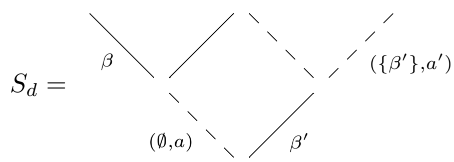
$$\mathrm{Con}_{S_d} \equiv \{R \subseteq (S_d)_E \mid (\chi_d(R) \in \mathrm{Con}_E) \wedge (\sigma_d{}^{\rightarrow}(R) \in \mathrm{Con}_A)\}$$

then $X \in \mathrm{Con}_{S_d}$ but there is no consistent superset which is down-closed.

Example 0.11.



$d \equiv [\emptyset \mapsto \emptyset, \{\beta\} \mapsto \{a, b\}, \{\beta'\} \mapsto \{b\}]$. Then



So $f_d{}^{\rightarrow}(\{\beta\}) = ([a']^A)^+$ but $d([\{\beta\}]^E)$ has no extension containing a'. Hence, in the definition of N_d we must include the constraint that if $(x, a) \in N_d$ then $d([x]^E) \cup \{a\} \in \mathrm{Con}_A$.

Lemma 0.12. *Let A and E be games such that the events of E are all positive. Let $d : \mathcal{C}(E) \to \mathcal{C}(A)$, $\sigma : S \to A$ a map of games, and $\phi : E \to S^+$ an map of games such that $d_\sigma \circ (\phi^{\rightarrow} \restriction \mathcal{C}(E)) = d$. Then $\beta(d, E, A)$.*

Proof. We show $\beta(d_\sigma, S^+, A)$. Fix $a \in S^+_E$ and $y \in \mathcal{C}(S^+)$. Assume $y \overset{a}{\longrightarrow\!\!\subset}_{S^+}$.

$$\begin{aligned}\sigma^{\rightarrow}([y \cup \{a\}]^S) &= \sigma^{\rightarrow}([y]^S \cup [a)^S \cup \{a\}) \\ &= \sigma^{\rightarrow}([y]^S) \cup \sigma^{\rightarrow}([a)^S) \cup \{\sigma(a)\}\end{aligned}$$

$\sigma(a) \notin \sigma^{\rightarrow}([y]^S)$ and $\sigma(a) \notin \sigma^{\rightarrow}([a)^S)$. Hence

$$d_\sigma(y \cup \{a\}) \setminus \{\sigma(a)\} = \sigma^{\rightarrow}([y \cup \{a\}]^S) \setminus \{\sigma(a)\} = \sigma^{\rightarrow}([y]^S) \cup \sigma^{\rightarrow}([a)^S)$$

which is the union of two compatible configurations so $d_\sigma(y \cup \{a\}) \setminus \{\sigma(a)\} \in \mathcal{C}(A)$. Since $\sigma(a) \notin d_\sigma(y)$ it follows that $\sigma(a) \in d_\sigma(y \cup \{a\}) \setminus d_\sigma(y)$. If $e \in [a)^S$ then $e \lneq_S a$ and if e is positive then $e \lneq_{S^+} a$ so $e \in y$ and so $\sigma(e) \in d_\sigma(y)$. Hence

$$d_\sigma(y \cup \{a\}) \setminus \{\sigma(a)\} = d_\sigma(y) \cup \sigma^{\rightarrow}(([a)^S)^-)$$

In all, this shows $\beta(d_\sigma, S^+, A)$.

If we fix $a \in E_E$ and $y \in \mathcal{C}(E)$ and assume that $y \xrightarrow{a}\!\!\subset_E$ then $\phi^\rightarrow(y) \xrightarrow{\phi(a)}\!\!\subset_{S^+}$ hence,

$$(d_\sigma(\phi^\rightarrow(y) \cup \{\phi(a)\}) \setminus \{\sigma(a)\} \in \mathcal{C}(A)) \wedge (\sigma(a) \in (d_\sigma(\phi^\rightarrow(y) \cup \{\phi(a)\}) \setminus d_\sigma(\phi^\rightarrow(y))))$$

Since $d_\sigma \circ (\phi^\rightarrow \restriction \mathcal{C}(E)) = d$ then

$$(d(y \cup \{a\}) \setminus \{\sigma(a)\} \in \mathcal{C}(A)) \wedge (\sigma(a) \in (d(y \cup \{a\}) \setminus d(y)))$$

□

Lemma 0.13. *Let A and E be games such that the events of E are all positive. Let $d : \mathcal{C}(E) \to \mathcal{C}(A)$ and suppose $\beta(d, E, A)$ and d preserves unions and $d(\emptyset) = \emptyset$. Then $\sigma_d : S_d \to A$ is a receptive, negative innocent map of games. And ϕ_d is an isomorphism of games. Also, $d_{\sigma_d} \circ ({\phi_d}^\rightarrow \restriction \mathcal{C}(E)) = d$.*

Under these conditions,

Lemma 0.14. *Let $y \in \mathcal{C}(E)$ and $\beta \in d(y)$ with $\mathrm{pol}(\beta) = +$ then $\exists b \in y \,.\, \beta = f_d(b)$.*

Proof. sketch: induction on $y = \emptyset \xrightarrow{b_0}\!\!\subset \ldots \xrightarrow{b_n}\!\!\subset y$. □

Lemma 0.15. *If $x, y \in \mathcal{C}(E)$ and $x \subseteq y$ then $d(x) \subseteq d(y)$.*

Lemma 0.16. *If $b \in y \in \mathcal{C}(E)$ then $f_d(b) \in d(y)$.*

Lemma 0.17. *Let $e, e' \in x \in \mathcal{C}(E)$. If $f_d(e) = f_d(e')$ then $e = e'$.*

Lemma 0.18. *If $(b, (x, a)) \in \leq_{S_d}$ then $((x, a), b) \notin \leq_{S_d}$.*

Proof. Suppose $(b, (x, a)) \in \leq_{S_d}$. Then $b \in [x]^E$ so there exists $b \leq_E b' \in x$ and $(x, a) \in N_d$ so ${f_d}^\rightarrow(x) = ([a]^A)^+$ so $f_d(b') \leq_A a$. Since $[b')^E \xrightarrow{b'}\!\!\subset_E$ then $d([b']^E) \setminus \{f_d(b')\} \in \mathcal{C}(A)$ by $\beta(d, E, A)$. If $((x, a), b) \in \leq_{S_d}$ then $a \in d([b]^E) \subseteq d([b']^E)$ by lemma 0.15 then $a \in d([b']^E) \setminus \{f_d(b')\}$, which contradicts $f_d(b') \leq_A a$. □

Lemma 0.19. *Let $x \in \mathcal{C}(E)$, $a \in M_d$ with $([a]^A)^+ \subseteq d(x)$ and $d(x) \cup \{a\} \in \mathrm{Con}_A$ then*

$$(({f_d}^\leftarrow(([a]^A)^+)) \cap x, a) \in N_d$$

Proof.

- $([a]^A)^+ = {f_d}^\rightarrow(({f_d}^\leftarrow(([a]^A)^+)) \cap x)$. By definition we have

$${f_d}^\rightarrow(({f_d}^\leftarrow(([a]^A)^+)) \cap x) \subseteq ([a]^A)^+$$

Let $\beta \in ([a]^A)^+$. Then $\beta \in d(x)$ by assumption. Then by lemma 0.14 there exists $b \in x$ such that $\beta = f_d(b)$. Then $b \in {f_d}^\leftarrow(([a]^A)^+)$ since $f_d(b) \in ([a]^A)^+$ so $\beta \in {f_d}^\rightarrow(({f_d}^\leftarrow(([a]^A)^+)) \cap x)$. Hence

$$([a]^A)^+ \subseteq {f_d}^\rightarrow(({f_d}^\leftarrow(([a]^A)^+)) \cap x)$$

- $(f_d{}^{\leftarrow}(([a]^A)^+)) \cap x \in \mathrm{Con}_E$. Since $x \in \mathrm{Con}_E$.
- $[(f_d{}^{\leftarrow}(([a]^A)^+)) \cap x]^E \subseteq x$ so $d([(f_d{}^{\leftarrow}(([a]^A)^+)) \cap x]^E) \cup \{a\} \subseteq d(x) \cup \{a\} \in \mathrm{Con}_A$ so $d([(f_d{}^{\leftarrow}(([a]^A)^+)) \cap x]^E) \cup \{a\} \in \mathrm{Con}_A$.

□

Lemma 0.20. *Let $b \in E_E$. Then $\sigma_d{}^{\rightarrow}([b]^{S_d}) = d([b]^E)$*

Proof.

- Let $e \in d([b]^E)$.
  - $\mathrm{pol}(e) = +$. Then $e = f_d(b')$ for some $b' \in [b]^E$ by lemma 0.14 so $b' \leq_{S_d} b$ and $e = \sigma_d(b')$. Hence $e \in \sigma_d{}^{\rightarrow}(\{c \in (S_d)_E \mid c \leq_{S_d} b\})$.
  - $\mathrm{pol}(e) = -$. Let $x \equiv (f_d{}^{\leftarrow}(([e]^A)^+)) \cap [b]^E$. By lemma 0.19, $(x, e) \in N_d$. Since, in addition, $(x, e) \leq_{S_d} b$ then

$$e \in \sigma_d{}^{\rightarrow}(\{c \in (S_d)_E \mid c \leq_{S_d} b\})$$

- Let $e \in \sigma_d{}^{\rightarrow}(\{c \in (S_d)_E \mid c \leq_{S_d} b\})$. Then $e = \sigma_d(c)$ with $c \in (S_d)_E$ and $c \leq_{S_d} b$.
  - $\mathrm{pol}(c) = +$. Then $c \leq_E b$ so $c \in [b]^E$. By lemma 0.16 $f_d(c) = \sigma_d(c) = e \in d([b]^E)$.
  - $\mathrm{pol}(c) = -$. Then $c \in N_d$ so $c = (x, a)$ for some $a \in M_d$. Since $(x, a) \leq_{S_d} b$ then $a = \sigma_d(c) \in d([b]^E)$ by definitions.

□

Lemma 0.21. *Let $(x, a) \in N_d$. Then $\sigma_d{}^{\rightarrow}([(x, a)]^{S_d}) = [a]^A \cup d([x]^E) \in \mathcal{C}(A)$*

Proof.

- Let $c \in \sigma_d{}^{\rightarrow}([(x, a)]^{S_d})$
  - $a' = c = \sigma_d((x', a'))$ for $(x', a') \leq_{S_d} (x, a)$. Then either $a' \leq_A a$ in which case $a' \in [a]^A$ or there exists b' such that $(x', a') \leq_{S_d} b'$ and $b' \leq_{S_d} (x, a)$ in which case $b' \in [x]^E$ so $[b']^E \subseteq [x]^E$ and $a' \in d([b']^E)$ so since d is monotone, $a' \in d([x]^E)$.
  - $\beta = f_d(b) = \sigma_d(c)$ for $b \leq_{S_d} (x, a)$. Then $b \in [x]^E$ so $f_d(b) \in d([x]^E)$ by lemma 0.16.
-
  - Suppose $a' \in ([a]^A)^-$. Let $x' \equiv f_d{}^{\leftarrow}(([a']^A)^+) \cap [x]^E$. By lemma 0.19, $(x', a') \in N_d$ ($([a']^A)^+ \subseteq ([a]^A)^+ = f_d{}^{\rightarrow}(x) \subseteq d([x]^E)$ by lemma 0.16). Let $b' \in x'$ then $f_d(b') \in ([a']^A)^+ \subseteq ([a]^A)^+ = f_d{}^{\rightarrow}(x)$ so there exists $b \in x$ such that $f_d(b) = f_d(b')$. Since $b, b' \in [x]^E$ then by lemma 0.17, $b = b' \in x$. Hence $x' \subseteq x$ so $(x', a') \leq_{S_d} (x, a)$ so $a' \in \sigma_d{}^{\rightarrow}([(x, a)]^{S_d})$.
  - Let $\beta \in ([a]^A)^+$. Since $(x, a) \in N_d$ then $f_d{}^{\rightarrow}(x) = ([a]^A)^+$ so there exists $b' \in x$ such that $\beta = \sigma_d(b')$. So $b' \leq_{S_d} (x, a)$ and $\beta \in \sigma_d{}^{\rightarrow}([(x, a)]^{S_d})$.
  - Suppose $\beta \in d([x]^E)^+$ then $\beta = f_d(b)$ for some $b \in [x]^E$ so $b \leq_{S_d} (x, a)$ so $\sigma_d{}^{\rightarrow}([(x, a)]^{S_d})$.

- Suppose $a' \in d([x]^E)^-$. If $x = \emptyset$ then $d([x]^E) = \emptyset$ by assumption. Else, since d preserves unions then $a' \in d([b']^E)$ for some $b' \in x$. Define $x' \equiv f_d{}^{\leftarrow}(([a']^A)^+) \cap [b']^E$ then by lemma 0.19, $(x', a') \in N_d$. Then $(x', a') \leq_{S_d} b'$ and $b' \leq_{S_d} (x, a)$. So, by definition, $(x', a') \leq_{S_d} (x, a)$.

- Since $(x, a) \in N_d$ then $d([x]^E) \cup \{a\} \in \mathrm{Con}_A$ so

$$[d([x]^E) \cup \{a\}]^A = [a]^A \cup d([x]^E) \in \mathcal{C}(A)$$

□

Lemma 0.22. *Let $b \in E_E$. Then $\chi_d([b]^{S_d}) = [b]^E$*

Proof. If $b' \in \chi_d([b]^{S_d})$ then either $b' \in [b]^E$ or $b' \in x$ and $(x, a) \leq_{S_d} b$ so $b' \in x \subseteq [b]^E$. If $b' \in [b]^E$ then $b' \in [b]^{S_d}$ so $b' \in \chi_d([b]^{S_d})$. □

Lemma 0.23. *Let $(x, a) \in N_d$. Then $\chi_d([(x, a)]^{S_d}) = [x]^E$*

Proof.

- $b \in \chi_d([(x, a)]^{S_d})$.
  - $b \leq_{S_d} (x, a)$ so $b \in [x]^E$.
  - $b \in x'$ and $(x', a') \leq_{S_d} (x, a)$
    - $x' \subseteq x$ so $b \in [x]^E$.
    - There exists $b' \in P_d$ such that $(x', a') \leq_{S_d} b' \leq_{S_d} (x, a)$ so $x' \subseteq [b']^E$ and $b' \in [x]^E$ so $b \in [x]^E$.
- $b \in [x]^E$. Then $b \leq_{S_d} (x, a)$ so $b \in \chi_d([(x, a)]^{S_d})$.

□

Lemma 0.24. *Let $y \in \mathcal{C}(E)$ then $d(y) = \sigma_d{}^{\rightarrow}([y]^{S_d})$*

Proof. By induction on $|y|$. True for $y = \emptyset$ since $d(\emptyset) = \emptyset$. Now suppose $|y| = n + 1$ then there exists b such that $y' \xrightarrow{b}\!\!\subset_E y$. Then

$$\begin{aligned}\sigma_d{}^{\rightarrow}(\{c \in (S_d)_E \mid \exists b \in y \,.\, c \leq_{S_d} b\}) &= \sigma_d{}^{\rightarrow}(\{c \in (S_d)_E \mid \exists b' \in y' \,.\, c \leq_{S_d} b'\}) \cup \sigma_d{}^{\rightarrow}([b]^{S_d}) \\ &= d(y') \cup d([b]^E) = d(y' \cup [b]^E) = d(y' \cup \{b\}) = d(y)\end{aligned}$$

using lemma 0.20 and assumptions. □

Lemma 0.25. *$(z, a), (z', a) \in N_d$ with $z, z' \subseteq x \in \mathcal{C}(E)$ then $z = z'$*

Proof. We have $f_d{}^{\rightarrow}(z) = f_d{}^{\rightarrow}(z') = ([a]^A)^+$ since $(z, a), (z', a) \in N_d$. By lemma 0.17, f_d is injective on x so $z' = z$. □

Lemma 0.26. *$\sigma_d : S_d \to A$ is a map of games.*

Proof. First, we show that S_d is an event structure.

- proof that $\leq_{S_d}$ is a partial order. $\leq_{S_d}$ is reflexive.

- If $b \in P_d$ then $(b, b) \in \leq_{S_d}$.
- If $(x, a) \in N_d$ then $((x, a), (x, a)) \in \leq_{S_d}$

Transitivity:

- $(b_1, b_2), (b_2, b_3) \in \leq_{S_d}$. Then $(b_1, b_3) \in \leq_{S_d}$.
- $(b_1, b_2), (b_2, (x, a)) \in \leq_{S_d}$. So $b_1 \leq_E b_2$ and $b_2 \in [x]^E$ so $b_1 \in [x]^E$ so $(b_1, (x, a)) \in \leq_{S_d}$.
- $(b, (x, a)), ((x, a), b') \in \leq_{S_d}$. $b \in [x]^E$ and $x \subseteq [b']^E$ so $b \leq_E b'$.
- $(b, (x, a)), ((x, a), (x', a')) \in \leq_{S_d}$. Then $b \in [x]^E$ and we have the following cases:
  * $x \subseteq x'$ and $a \leq_A a'$. So $b \in [x']^E$ so $(b, (x', a')) \in \leq_{S_d}$.
  * There exists b' such that $((x, a), b'), (b', (x', a')) \in \leq_{S_d}$. Then by previous case $(b, b') \in \leq_{S_d}$ so $(b, (x', a')) \in \leq_{S_d}$ by previous case.
- $((x, a), b), (b, b') \in \leq_{S_d}$. Then $a \in d([b]^E)$ and $x \subseteq [b]^E$ and $b \leq_E b'$ so $[b]^E \subseteq [b']^E$ so $d([b]^E) \subseteq d([b']^E)$ so $a \in d([b']^E)$ and $x \subseteq [b']^E$. So $((x, a), b') \in \leq_{S_d}$.
- $((x_1, a_1), (x_2, a_2)), ((x_2, a_2), b) \in \leq_{S_d}$. Then $x_2 \subseteq [b]^E$ and $a_2 \in d([b]^E)$.
  * $x_1 \subseteq x_2$ and $a_1 \leq_A a_2$ then $x_1 \subseteq [b]^E$ and $a_1 \in d([b]^E)$ so $((x_1, a_1), b) \in \leq_{S_d}$.
  * There exists b' such that $((x_1, a_1), b'), (b', (x_2, a_2)) \in \leq_{S_d}$ so $(b', b) \in \leq_{S_d}$ so $((x_1, a_1), b) \in \leq_{S_d}$.
- $((x_1, a_1), (x_2, a_2)), ((x_2, a_2), (x_3, a_3)) \in \leq_{S_d}$.
  * $x_1 \subseteq x_2$ and $a_1 \leq_A a_2$ and $x_2 \subseteq x_3$ and $a_2 \leq_A a_3$ then $x_1 \subseteq x_3$ and $a_1 \leq_A a_3$ so $((x_1.a_1), (x_3, a_3)) \in \leq_{S_d}$.
  * There exists b such that $((x_1, a_1), b), (b, (x_2, a_2)) \in \leq_{S_d}$ so $(b, (x_3, a_3)) \in \leq_{S_d}$ so $((x_1.a_1), (x_3, a_3)) \in \leq_{S_d}$.
  * There exists b such that $((x_2, a_2), b), (b, (x_3, a_3)) \in \leq_{S_d}$ so $((x_1, a_1), b) \in \leq_{S_d}$ so $((x_1.a_1), (x_3, a_3)) \in \leq_{S_d}$.
- $((x, a), b), (b, (x', a')) \in \leq_{S_d}$. Immediate from the definition of $\leq_{S_d}$.

For anti-symmetry,

- $(b_1, b_2), (b_2, b_1) \in \leq_{S_d}$ so $b_1 \leq_E b_2$ and $b_2 \leq_E b_1$ so $b_1 = b_2$.
- $(b, (x, a)), ((x, a), b) \in \leq_{S_d}$. Impossible by lemma 0.18
- $((x_1, a_1), (x_2, a_2)), ((x_2, a_2), (x_1, a_1)) \in \leq_{S_d}$. Splits into
  * $x_1 \subseteq x_2$ and $a_1 \leq_A a_2$ and $x_2 \subseteq x_1$ and $a_2 \leq_A a_1$ so $(x_1, a_1) = (x_2, a_2)$.
  * There exists b such that $((x_1, a_1), b), (b, (x_2, a_2)) \in \leq_{S_d}$. Then by transitivity, $(b, (x_1, a_1)) \in \leq_{S_d}$, which is impossible by lemma 0.18.

    * There exists b such that $((x_2, a_2), b), (b, (x_1, a_1)) \in \leq_{S_d}$. Then by transitivity, $(b, (x_2, a_2)) \in \leq_{S_d}$, which is impossible by lemma 0.18.

  - $((x, a), b), (b, (x, a)) \in \leq_{S_d}$. Impossible by lemma 0.18

- proof that $\{e' \in (S_d)_E \mid e' \leq_{S_d} e\}$ is finite

  $\{e' \in (S_d)_E \mid e' \leq_{S_d} e\} = \{b \in P_d \mid b \leq_{S_d} e\} \cup \{(x, a) \in N_d \mid (x, a) \leq_{S_d} e\}$

  - $e = b' \in P_d$. Then

    $$|\{b \in P_d \mid b \leq_{S_d} b'\}| = |\{b \in E_E \mid b \leq_E b'\}|$$

    which is finite and

    $\{(x, a) \in N_d \mid (x, a) \leq_{S_d} e\} = \{(x, a) \in N_d \mid x \subseteq [b']^E \wedge a \in d([b']^E)\}$

    which is finite since configurations in A are finite.

  - $e = (x', a') \in N_d$. Then

    $$\{b \in P_d \mid b \leq_{S_d} (x', a')\} = \{b \in P_d \mid b \in [x']^E\}$$

    which is finite since x' is finite.

    $$\{(x, a) \in N_d \mid (x, a) \leq_{S_d} (x', a')\} = \{(x, a) \in N_d \mid x \subseteq x' \wedge a \leq_A a'\} \cup \{(x, a) \in N_d \mid \exists b \in P_d \,.\, (x, a) \leq_{S_d} b \wedge b \leq_{S_d} (x', a')\}$$

    The first set is finite and the second:

    $$|\{(x, a) \in N_d \mid \exists b \in P_d \,.\, (x, a) \leq_{S_d} b \wedge b \leq_{S_d} (x', a')\}| \leq \sum_{b \in x'} |\{(x, a) \in N_d \mid (x, a) \leq_{S_d} b\}|$$

    which is a finite sum of finite sets.

- Proof that $\{e\} \in \mathrm{Con}_{S_d}$ for $e \in (S_d)_E$.

  - $e = b \in P_d$. $\chi_d([b]^{S_d}) = [b]^E \in \mathcal{C}(E)$ and $\sigma_d{}^{\rightarrow}([b]^{S_d}) = d([b]^E) \in \mathcal{C}(A)$ by lemmas 0.20 and 0.22.
  - $e = (x, a) \in N_d$. $\chi_d([(x, a)]^{S_d}) = [x]^E \in \mathcal{C}(E)$ and $\sigma_d{}^{\rightarrow}([(x, a)]^{S_d}) \in \mathcal{C}(A)$ lemmas 0.21 and 0.23.

- Proof that for $X, Y \subseteq (S_d)_E$ if $X \subseteq Y \in \mathrm{Con}_{S_d}$ then $X \in \mathrm{Con}_{S_d}$.

  - We have $[X]^{S_d} \subseteq [Y]^{S_d}$ so $\sigma_d{}^{\rightarrow}([X]^{S_d}) \subseteq \sigma_d{}^{\rightarrow}([Y]^{S_d}) \in \mathcal{C}(A)$ so $\sigma_d{}^{\rightarrow}([X]^{S_d}) \in \mathrm{Con}_A$. We have

    $$[X]^{S_d} = \bigcup_{e \in [X]^{S_d}} [e]^{S_d} \quad \text{so} \quad \sigma_d{}^{\rightarrow}([X]^{S_d}) = \bigcup_{e \in [X]^{S_d}} \sigma_d{}^{\rightarrow}([e]^{S_d})$$

    By lemmas 0.20 and 0.21, for $e \in [X]^{S_d}$, $\sigma_d{}^{\rightarrow}([e]^{S_d}) \in \mathcal{C}(A)$ and is bounded above by the configuration $\sigma_d{}^{\rightarrow}([Y]^{S_d})$ so $\sigma_d{}^{\rightarrow}([X]^{S_d}) \in \mathcal{C}(A)$.

- χ_d is monotone and preserves unions, like $\sigma_d{}^{\rightarrow}$, so there is a nearly identical proof using lemmas 0.22 and 0.23.

- Proof that for $e' \in X \in \mathrm{Con}_{S_d}$ if $e \leq_{S_d} e'$ then $X \cup \{e\} \in \mathrm{Con}_{S_d}$. Follows immediately from definitions since $[X \cup \{e\}]^{S_d} = [X]^{S_d}$.

Now we show that σ_d is a map of games $S_d \to A$.

- If $X \in \mathcal{C}(S_d)$ then $X \in \mathrm{Con}_{S_d}$ and $[X]^{S_d} = X$ so $\sigma_d{}^{\rightarrow}(X) \in \mathcal{C}(A)$.
- Let $e, e' \in X \in \mathcal{C}(S_d)$ with $\sigma_d(e) = \sigma_d(e')$. We will show $e = e'$.
  - $e = b, e' = b' \in P_d$. Then $f_d(b) = f_d(b')$ for $b, b' \in \chi_d(X) \in \mathcal{C}(E)$. So $b = b'$ by lemma 0.17.
  - $e = (x, a), e' = (x', a) \in N_d$. Since $x, x' \subseteq \chi_d(X) \in \mathcal{C}(E)$ by definition of χ_d then by lemma 0.25, $x = x'$.
- σ_d preserves polarities. By definitions.

□

Lemma 0.27. *If $X \in \mathcal{C}(S_d)$ then $\chi_d(X) = X^+$*

Proof.
- $b \in \chi_d(X)$. Then either $b \in X$ or $b \in x$ and $(x, a) \in X$ so $b \leq_{S_d} (x, a)$ so $b \in X$.
- $b \in X^+$. Then $b \in \chi_d(X)$ by definition.

□

Lemma 0.28. *$X \stackrel{(z,a)}{\longrightarrow\!\!\!\subset}_{S_d}$ then $z \subseteq \chi_d(X)$.*

Proof. Let $b \in z$ then $b \in [z]^E$ so $b \leq_{S_d} (z, a)$ so $b \in X$. Since $\mathrm{pol}(b) = +$ then $b \in \chi_d(X)$. □

Lemma 0.29. *σ_d is receptive.*

Proof. Let $X \in \mathcal{C}(S_d)$ and $\sigma_d{}^{\rightarrow}(X) \stackrel{a}{\longrightarrow\!\!\!\subset}_A$. Let $x \equiv \chi_d(X)$. Then $x \in \mathcal{C}(E)$. By lemma 0.27, $x \subseteq X$ so $[x]^{S_d} \subseteq X$ so $\sigma_d{}^{\rightarrow}([x]^{S_d}) = d(x) \subseteq \sigma_d{}^{\rightarrow}(X)$ so $d(x) \cup \{a\} \subseteq \sigma_d{}^{\rightarrow} \cup \{a\} \in \mathcal{C}(A)$ so $d(x) \cup \{a\} \in \mathrm{Con}_A$. Now

$([a])^+ \subseteq (\sigma_d{}^{\rightarrow}(X))^+$	if $\beta \in ([a])^+$ then $\beta \lneq_A a$ so $\beta \in \sigma_d{}^{\rightarrow}(X)$
$\subseteq f_d{}^{\rightarrow}(x)$	if $\beta \in (\sigma_d{}^{\rightarrow}(X))^+$ then $\exists b \in X^+ \,.\, \beta = \sigma_d(b) = f_d(b)$ and $X^+ = x$ by lemma 0.27
$\subseteq d(x)$	by lemma 0.16

so $d(x) \subseteq^- d(x) \cup [a]^A \in \mathcal{C}(A)$. Hence $a \in M_d$. Let $z \equiv (f_d{}^{\leftarrow}(([a]^A)^+)) \cap x$. By lemma 0.19, $(z, a) \in N_d$.

Now we show that $X \stackrel{(z,a)}{\longrightarrow\!\!\!\subset}_{S_d}$.

- $X \cup \{(z, a)\}$ is $\leq_{S_d}$-down-closed.

- if $b \leq_{S_d} (z, a)$ then $b \in [z]^E \subseteq x \subseteq X$ so $b \in X$.
- if $(z', a') \leq_{S_d} (z, a)$ then
  - $z' \subseteq z$ and $a' \leq_A a$.
    - $a' = a$. Since $z' \subseteq z \subseteq x \in \mathcal{C}(E)$ then, by lemma 0.25, $z = z'$. So $(z', a') = (z, a)$.
    - $a' \lneq_A a$. Since ${\sigma_d}^{\rightarrow}(X) \stackrel{a}{-\!\!\!\subset}_A$ then $a' \in {\sigma_d}^{\rightarrow}(X)$ so there exists $(z'', a') \in X$. We have $z'' \subseteq \chi_d(X) = x$ by definition of χ_d. We also have $z' \subseteq x \in \mathcal{C}(E)$ then, by lemma 0.25, $z' = z''$. So $(z', a') \in X$.
  - There exists $b \in P_d$ such that $(z', a') \leq_{S_d} b \leq_{S_d} (z, a)$ so $b \in X$ by previous item so $(z', a') \in X$ since $X \in \mathcal{C}(S_d)$.

- $X \cup \{(z, a)\} \in \mathrm{Con}_{S_d}$. $\chi_d(X \cup \{(z, a)\}) = \chi_d(X) \cup z = \chi_d(X)$ since $z \subseteq x$ and $\chi_d(X) \in \mathcal{C}(E)$ since $X \in \mathcal{C}(S_d)$. ${\sigma_d}^{\rightarrow}(X \cup \{(x, a)\}) = {\sigma_d}^{\rightarrow}(X) \cup \{a\} \in \mathcal{C}(A)$ since ${\sigma_d}^{\rightarrow}(X) \stackrel{a}{-\!\!\!\subset}_A$.
- $(z, a) \notin X$. Otherwise, $a \in {\sigma_d}^{\rightarrow}(X)$ which would contradict ${\sigma_d}^{\rightarrow}(X) \stackrel{a}{-\!\!\!\subset}_A$.

Finally, we show uniqueness. Suppose $X \stackrel{(z', a)}{-\!\!\!\subset}_{S_d}$. By lemma 0.28, $z, z' \subseteq x \in \mathcal{C}(E)$. By lemma 0.25, $z = z'$. ◻

Lemma 0.30. *σ_d is negative innocent.*

Proof. Suppose negative innocence fails; i.e. that $e \rightarrow\!\!\!\triangleright_{S_d} (z, a)$ but not $\sigma_d(e) \rightarrow\!\!\!\triangleright_A a$. Then for $X \equiv [e)^{S_d}$, we have $X \stackrel{e}{-\!\!\!\subset}_{S_d} \stackrel{(z,a)}{-\!\!\!\subset}_{S_d}$. So ${\sigma_d}^{\rightarrow}(X) \stackrel{\sigma_d(e)}{-\!\!\!\subset}_A \stackrel{a}{-\!\!\!\subset}_A$. Since not $\sigma_d(e) \rightarrow\!\!\!\triangleright_A a$ then ${\sigma_d}^{\rightarrow}(X) \stackrel{a}{-\!\!\!\subset}_A$. By receptivity, there exists $(z', a) \in N_d$ such that $X \stackrel{(z', a)}{-\!\!\!\subset}_{S_d}$. We have $z' \subseteq \chi_d(X) \subseteq \chi_d(X \cup \{e\})$ and also $z \subseteq \chi_d(X \cup \{e\})$ by lemma 0.28. By lemma 0.25, $z = z'$. Hence $X \stackrel{(z,a)}{-\!\!\!\subset}_{S_d}$. This contradicts $e \rightarrow\!\!\!\triangleright_{S_d} (z, a)$. ◻